Recent advances in description of few two-component fermions

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Abstract

Overview of the recent advances in description of the few two-component fermions is presented. The model of zero-range interaction is generally considered to discuss the principal aspects of the few-body dynamics. Particular attention is paid to detailed description of two identical fermions of mass m and a distinct particle of mass m_1: it turns out that two $L^P = 1^-$ three-body bound states emerge if mass ratio m/m_1 increases up to the critical value $\mu_c \approx 13.607$, above which the Efimov effect takes place. The topics considered include rigorous treatment of the few-fermion problem in the zero-range interaction limit, low-dimensional results, the four-body energy spectrum, crossover of the energy spectra for m/m_1 near μ_c, and properties of potential-dependent states. At last, enlisted are the problems, whose solution is in due course.

PACS numbers:

I. INTRODUCTION

In the recent years, properties of multi-component ultra-cold quantum gases, including binary Fermi-Bose [1–3] and Fermi [4–6] mixtures and impurities embedded in a quantum gas [7–9] are under thorough experimental and theoretical investigation. In this respect, the low-energy few-body dynamics in two-species mixtures has attracted much attention. In particular, study of the energy spectrum and low-energy scattering of few two-component particles gives insight into the role of few-body processes in the many-body dynamics. One should also mention the reactions with negative atomic and molecular ions [10, 11] and the two-component model for the three-body recombination near the Feshbach resonance [12].

The aim of the paper is to present an overview of the recent advances in description of few ultra-cold two-component particles; it is assumed that identical particles are fermions interacting with a distinct particle via the s-wave potential, whereas interaction of identical fermions is forbidden. For the sake of generality, in some cases it is suitable to consider also a system of few non-interacting identical bosons and a distinct particle. It is worthwhile to mention that the p-wave interactions between fermions may be also important, in particular, an infinite number of the 1^+ bound states was predicted [13] for three identical fermions.

To investigate the principal aspects of the few-body dynamics, it is natural to use the limit of zero-range two-body interaction, which allows one to obtain universal description of the few-body properties. In this respect, it is necessary to formulate rigorously the few-body problem in the zero-range interaction limit, which is an interesting problem in itself, especially if the identical particles are fermions.

The zero-range model is suitably defined by imposing the boundary condition at the zero inter-particle distance. Thus, the interaction depends on a single parameter, e. g., the two-body scattering length a, which can be chosen as a length scale. As a result, the energy scale is $1/a^2$, and the only remaining parameter is mass ratio m/m_1, where m and m_1 denote masses of identical and distinct particles, respectively. The units $\hbar = |a| = 2m/(1 + m/m_1) = 1$, for which the two-body binding energy $\varepsilon_2 = 1$, will be used throughout the paper.

II. UNIVERSAL PROPERTIES OF THREE TWO-COMPONENT FERMIONS

Investigation of two identical fermions and a distinct particle are of considerable interest for description of the multi-component ultra-cold gases. The three-body states of unit total angular momentum and negative parity ($L^P = 1^-$) are especially important in treatment of the low-energy processes [14, 15]. A major progress was achieved in [16], where it was shown that the zero-range model for sufficiently large mass ratio $m/m_1 > \mu_c$ does not provide unique description of the three two-component fermions. The unambiguous description of the three-body properties for $m/m_1 > \mu_c$ requires an additional parameter, which determines the wave function in the vicinity of the triple-collision point. As shown in [16], a number of three-body bound states is infinite and their energies differ by the scaling factor. Later on, properties of the three-body spectrum were discussed also in a number of subsequent papers [11, 17–20]. The critical mass ratio is determined by a solution of the transcendental equation for $\sin\omega = 1/(1 + m_1/m)$

$$\frac{\pi}{2}\sin^2\omega_c - \tan\omega_c + \omega_c = 0\,, \tag{1}$$

which gives $\omega_c \approx 1.19862376$ and $\mu_c = \sin\omega_c/(1 - \sin\omega_c) \approx 13.6069657$. Furthermore, a significant achievement in this area was the construction of zero-energy solution for three two-component fermions in the interval $0 \le m/m_1 \le \mu_c$ [14], which provides the analytical expression for the low-energy recombination rate. A complete description of both the energy spectrum and the elastic and inelastic low-energy scattering is discussed below.

A. Hyperradial equations

In the zero-range limit, the two-body interaction is defined by imposing the boundary condition at the zero inter-particle distance r

$$\lim_{r\to 0}\frac{\partial \ln(r\Psi)}{\partial r} = -\mathrm{sign}(a)\ . \tag{2}$$

The interaction introduced by means of the boundary condition is widely discussed in the literature [21–24].

Both qualitative and numerical results are obtained by using the solution of hyper-radial

equations (HREs) [25]

$$\left[\frac{d^2}{d\rho^2} - \frac{\gamma_n^2(\rho) - 1/4}{\rho^2} + E\right] f_n(\rho) - \sum_{m=1}^{\infty}\left[P_{mn}(\rho) - Q_{mn}(\rho)\frac{d}{d\rho} - \frac{d}{d\rho}Q_{mn}(\rho)\right] f_m(\rho) = 0\ , \quad (3)$$

whose terms $\gamma_n^2(\rho)$, $Q_{nm}(\rho)$, $P_{nm}(\rho)$ are derived analytically [26, 27]. In fact, the critical mass ratio μ_c can be determined from the condition $\gamma_1(0) = 0$ as the first-channel effective potential at small ρ takes the form $\left[\gamma_1^2(0) - 1/4\right]/\rho^2$, which implies that a number of the bound states is finite for $\gamma_1^2(0) > 0$ and infinite for $\gamma_1^2(0) < 0$.

B. Unit angular momentum

Thorough studies of three two-component fermions in the states of total angular momentum and parity $L^P = 1^-$ were conducted in [15]. Only negative parity is considered, since for the positive-parity states three particles do not interact by the s-wave zero-range potential. For the problem under consideration, the functions $\gamma_n(\rho)$ determining the effective potentials in (3) satisfy the transcendental equation

$$\rho\,\mathrm{sign}(a) = \frac{1-\gamma^2}{\gamma}\tan\gamma\frac{\pi}{2} - \frac{2}{\sin 2\omega}\frac{\cos\gamma\omega}{\cos\gamma\frac{\pi}{2}} + \frac{\sin\gamma\omega}{\gamma\sin^2\omega\cos\gamma\frac{\pi}{2}}\ . \quad (4)$$

In particular, taking the limit $\gamma_1(0) \to 0$ in Eq. (4), one obtains Eq. (1) that determine μ_c. From the solution of HREs (3) it follows that for $a > 0$ there are no bound states in the interval $0 < m/m_1 < \mu_1$, exactly one bound state exists in the interval $\mu_1 \le m/m_1 < \mu_2$, and two bound states exist in the interval $\mu_2 \le m/m_1 \le \mu_c$, where $\mu_1 \approx 8.17260$ and $\mu_2 \approx 12.91743$. The bound-state energies decrease with increasing mass ratio on the interval $0 < m/m_1 \le \mu_c$, reaching the finite values $E_1(\mu_c) \approx -5.8954$, $E_2(\mu_c) \approx -1.13764$ at the critical value $m/m_1 = \mu_c$, and follow a square-root dependence $E_i - E_i(\mu_c) \propto (\mu_c - m/m_1)^{1/2}$ near $m/m_1 = \mu_c$. The dependence of bound-state energies on mass ratio is illustrated in Fig. 1.

For mass ratio just below μ_i ($m/m_1 \lesssim \mu_1$ and $m/m_1 \lesssim \mu_2$), the relevant bound state turns to a narrow resonance, whose position E_i^r continues a linear mass-ratio dependence of the bound-state energy, $E_i^r + 1 \propto \mu_i - m/m_1$, whereas the width Γ_i depends quadratically, $\Gamma_i \propto (\mu_i - m/m_1)^2$.

Calculations at the three-body threshold reveal the two-hump structure of the mass-ratio dependencies for the elastic (2 + 1)-scattering cross section and the three-body recombination rate. The dependence of the recombination rate is in accordance with the analytical

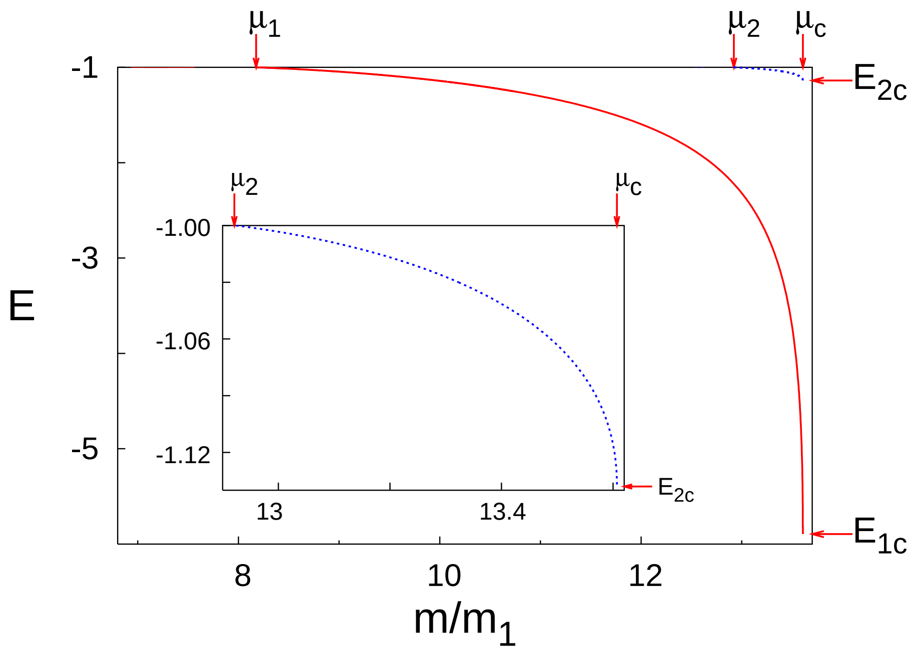

FIG. 1: Dependencies of the bound-state energies (in units of the two-body binding energy) on mass ratio m/m_1. The arrows mark mass ratios μ_i, for which the ith bound state emerges from the two-body threshold, the critical mass ratio μ_c, and the bound-state energies E_{ic} for $m/m_1 = \mu_c$. In the inset the excited-state energy is shown on a large scale.

expression [14]. The structure of both isotopic dependencies stems from the interference of the incoming and outgoing waves in the $2 + 1$ channel; the effect of interference is connected with emergence of two three-body bound states due to the deepening of the effective potential with increasing mass ratio. A detailed explanation of the mass-ratio dependence can be found in [15].

For negative scattering length $a < 0$ there are no three-body bound states in the interval $0 < m/m_1 \le \mu_c$.

C. Arbitrary angular momentum

To obtain the comprehensive description of three two-component particles, it is necessary to study the three-body properties for arbitrary total angular momentum. In this respect, the three-body rotational-vibrational spectrum was analysed in [28], where the identical particles were either fermions or bosons. It turns out that the properties of the three-body energy spectrum for arbitrary L resemble those for $L = 1$ [15].

Method of calculations for arbitrary L states in [28] was similar to that used for $L^P = 1^-$ states in [15], i. e., the system of HREs whose terms $\gamma_n^2(\rho)$, $Q_{nm}(\rho)$, $P_{nm}(\rho)$ are derived analytically [26, 27] was solved. By analysing the first-channel effective potential $(\gamma_1^2(\rho) - 1/4)/\rho^2$, it was shown in [28] that the bound states exist if either the identical particles are fermions and L is odd or the identical particles are bosons and L is even. Thus, it is suitable to treat jointly both bosonic problem for even L and fermionic problem for odd L. The most discussed feature [16, 17, 19, 20] of the three-body problem under consideration is an appearance of the infinite number of bound states for sufficiently large m/m_1. For each L, $\gamma_1^2(0)$ decreases with increasing m/m_1 and crosses zero at the critical value $\mu_c(L)$. Thus, the number of bound states is infinite for mass ratio above the critical value and finite below it. A set of critical mass ratios $\mu_c(L)$ are given in Table I for $L = 1 - 5$.

TABLE I: Upper part: Mass ratios $\mu_N(L)$ at which the N-th bound state arises and the critical values $\mu_c(L)$. Lower part: Bound-state energies $|E_{LN}(\mu_c(L))|$ at critical mass ratios.

L	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$	$N=6$	$\mu_c(L)$
1	7.9300	12.789	-	-	-	-	13.6069657
2	22.342	31.285	37.657	-	-	-	38.6301583
3	42.981	55.766	67.012	74.670	-	-	75.9944943
4	69.885	86.420	101.92	115.08	123.94	-	125.764635
5	103.06	123.31	142.82	160.64	175.48	185.51	187.958355
1	5.906	1.147	-	-	-	-	
2	12.68	1.850	1.076	-	-	-	
3	22.59	2.942	1.417	1.057	-	-	
4	35.59	4.392	1.920	1.273	1.049	-	
5	52.16	6.216	2.566	1.584	1.206	1.045	

It was shown that three particles are not bound for $a < 0$ and mass ratio in the interval $m/m_1 \le \mu_c(L)$. There is a finite number of bound states for $a > 0$ and $m/m_1 \le \mu_c(L)$. More precisely, three particles are unbound for sufficiently small m/m_1 and with increasing mass ratio the Nth bound state arises at $m/m_1 = \mu_N(L)$. The energies of each bound state $E_{LN}(m/m_1)$ monotonically decrease with increasing mass ratio, reaching the finite values at $\mu_c(L)$ and obeying the $[\mu_c(L) - m/m_1]^{1/2}$ dependence just below $\mu_c(L)$. For m/m_1 just

below the critical values $\mu_N(L)$, there are resonances, whose positions depend linearly and widths depend quadratically on the mass-ratio excess $\mu_N(L) - m/m_1$.

Reasonable estimates (up to a few percent accuracy) of bound-state energies for $a > 0$ were obtained numerically by using the one-channel approximation for the total wave function. Critical values of mass ratios $\mu_N(L)$ and three-body energies $E_{LN}(m/m_1)$ at $m/m_1 = \mu_c(L)$ are given in Table I. Accuracy of the one-channel approximation can be estimated by comparing the $L = 1$ values $\mu_N(1)$ and $E_{1N}[\mu_c(1)]$ from Table I with the precise values μ_1, μ_2, $E_1(\mu_c)$, and $E_2(\mu_c)$ given in Section II B. Notice that for $L = 3-5$ the uppermost bound states are loosely bound and appear very close to the corresponding critical values $\mu_c(L)$. Thus, taking into account an estimated accuracy of the approximation, one concludes that more careful calculation is necessary to describe these loosely bound states.

As it was shown in [15] for $L = 1$, an appearance of the three-body bound states with increasing mass ratio is intrinsically connected with the oscillating behaviour of the $2 + 1$ elastic-scattering cross section and the three-body recombination rate. Analogously, the dependence of the scattering amplitudes on mass ratio for higher L would exhibit the number of interference maxima which are related to the appearance of up to N_{max} bound states.

A reasonably good description of the energy spectrum is obtained within the framework of the quasi-classical approximation, which allows the asymptotic expression of bound-state energies $E_{LN}(m/m_1)$ to be derived for large L and m/m_1. As a result for $\mu_1(L) \le m/m_1 \le \mu_c(L)$ and $a > 0$, all the bound-state energies are described by the universal function of two scaled variables $\xi = (N - 1/2)/\sqrt{L(L+1)}$ and $\eta = \sqrt{m/[m_1 L(L+1)]}$. This scaling dependence is confirmed by the numerical calculations for $L > 2$ and is in good agreement even for small $L = 1, 2$. The universal description implies that $\mu_c(L) \approx 6.218(L+1/2)^2$ and $\mu_1(L) \approx 3.152(L + 1/2)^2$ for large L, while the number of vibrational states for given L is limited as $N_{max} \le 1.1\sqrt{L(L+1)} + 1/2$. More details are presented in paper [28].

D. Integral equations

Description of the two-component systems was obtained also by solving the momentum-space integral equations (generalised Skorniakov-Ter-Martirosian equations) in [29, 30]. These calculations generally confirm the results of [15, 28] and contain some additional details. One should mention that the values $\mu_N(1)$ calculated in [29] are in agreement with

those obtained in [15] for $L = 1$, while $\mu_N(L)$ from [29] for $L = 2-4$ slightly differ from the approximate values given in [28]. Besides, the characteristics of (2+1) zero-energy scattering for $L = 0-3$ were obtained in [29]. These calculations provide an additional evidence for the appearance of three-body bound states, in particular, the P-wave scattering volume diverges exactly at m/m_1 tending to the critical values μ_1 and μ_2. The mass-ratio dependence of the elastic $(2+1)$-scattering cross sections for different energies below the three-body threshold was studied in [30]. In addition, the momentum-space integral equations were applied to solution of different few-body problems [31–35]. More details of these calculations are given in Section VI.

E. Two-dimensional problem

Study of three two-component particles with zero-range interactions confined in two dimensions was performed in [36], where the mass-ratio dependence of the three-body energies and a set of critical values m/m_1, at which bound states emerge, were obtained by solving the momentum-space integral equations. Similar to 3D problem [28], the bound states exist if either the identical particles are fermions and L is odd or the identical particles are bosons and L is even; the binding energies monotonically increase with increasing mass ratio. Calculation [36] shows that for $L = 0$ two bosons and the third particle are bound for any mass ratio, while the second and third bound state appear at $m/m_1 \approx 1.77$ and $m/m_1 \approx 8.341$. Likewise, for $L = 1$ two fermions and the third particle are bound for $m/m_1 \geq 3.34$, while the second and third bound state appear at $m/m_1 \approx 10.41$ and $m/m_1 \approx 20.85$. As in 3D problem [28], different rotational-vibrational states become quasi-degenerate for large L and m/m_1.

More general problem for three two-component particles was considered in paper [37], where all three zero-range interactions were taken into account. In particular case of two noninteracting identical particles, a set of mass-ratio values, at which the $L = 0$ three-body bound states emerge, are consistent with those found in [36]. Furthermore, the energy spectrum of three 2D particles for different combinations of all masses and interaction strengths was considered in [38]. The S- and P-wave elastic $(2 + 1)$-scattering and the three-body recombination in 2D two-component mixtures were studied for mass ratio corresponding to both ^{6}Li–^{40}K and equal-mass particles [39]. A number of 2D bound-state and scattering

properties for three and four particles were calculated in [40], where different combinations of equal-mass bosons and fermions were considered.

The transition from two dimensions to three dimensions was studied in a quasi-two-dimensional geometry by confining particles in a harmonic potential along one direction [41]. It was shown that P-wave energies of two identical fermions and one distinct particle smoothly evolve from $3D$ to $2D$ (with increasing confining frequency) for $m/m_1 \leq \mu_c$. Correspondingly, the mass ratio, at which the three-body bound states emerges, increases from $2D$ value 3.33 [36] to $3D$ value 8.17260 [15]. In addition, it was estimated in [41] that in $2D$ limit three identical fermions and one distinct particle are bound for $m/m_1 > 5$.

F. One-dimensional problem

Properties of three two-component particles confined in one dimension with contact (δ-function) interactions were considered in [42]. It is assumed that attractive interaction of strength $\lambda < 0$ acts between each of two identical particles (either bosons or fermions) and a distinct one, while the strength of interaction between the identical particles is arbitrary λ_1.

The three-body energy spectrum and the scattering length A for collision of a bound pair off the third particle were calculated for two values $\lambda_1 = 0$ and $\lambda_1 \to \infty$ of the even-parity bosonic problem. Two sets of mass-ratio values, at which the three-body bound states arise and at which $A = 0$, were calculated both for $\lambda_1 = 0$ and $\lambda_1 \to \infty$. It is important to recall that the bosonic problem for $\lambda_1 \to \infty$ is exactly equivalent to the fermionic problem, for which the contact interaction between fermions is absent ($\lambda_1 = 0$). For the odd-parity states it was shown that three particles are not bound and the $(2+1)$ - scattering length for $\lambda_1 = 0$ was calculated.

In addition, few analytical results were presented, in particular, it was shown that exactly one bound state of three equal-mass particles ($m/m_1 = 1$) exists for arbitrary λ_1. Next, for two light fermions (in the limit of $m/m_1 \to 0$) three particles are not bound for sufficiently large repulsive interaction between fermions, viz., for $\lambda_1/|\lambda|$ above the critical value ≈ 2.66735 [43], and exactly one bound state exists for $\lambda_1/|\lambda|$ below this value. To elucidate the general features of three one-dimensional particles, both numerical and analytical results were used to construct a schematic "phase" diagram, which shows the number of three-body

bound states and a sign of the $(2+1)$-scattering length A in the plane of the parameters m/m_1 and $\lambda_1/|\lambda|$.

G. Dimensional analysis

Contrary to the three-body problem in 3D, the 2D and 1D solutions remain regular near the triple-collision point even in the limit of zero-range two-body interaction; therefore, there is neither Thomas nor Efimov effect. As a result, it is not necessary to introduce an additional regularisation parameter and the low-energy three-body properties in 2D and 1D are completely determined by the two-body input.

The results of calculations [15, 28, 36, 42] give an opportunity to analyse the dependence of the three-body low-energy properties on the configuration-space dimension. It turns out that two identical fermions and a distinct particle are bound in $1D$ for $m/m_1 \geq 1$, in $2D$ for $m/m_1 \geq 3.33$ and in $3D$ for $m/m_1 \geq 8.17260$. Two identical non-interacting bosons and a distinct particle are bound in $1D$ and $2D$ for any mass ratio, the first excited state appears in $1D$ at $m/m_1 \approx 2.869539$ and in $2D$ at $m/m_1 \approx 1.77$, while in $3D$ the number of bound states is infinite. If the three-body binding energy exceeds the two-body binding energy, the production of the triatomic molecules becomes energetically more favourable than diatomic ones. For two identical fermions and a distinct particle it is justified if $m/m_1 > 49.8335$ in $1D$, $m/m_1 > 18.3$ in $2D$, and $m/m_1 > 12.69471$ in $3D$.

III. POTENTIAL-DEPENDENT STATES

Analysing the few-body properties for the small interaction range r_0 tending to zero, it is necessary to take into account two different length scales, r_0 and a ($r_0 << a$), which means that all the states should be classified as either universal ones, whose energies scale according to a^{-2} or potential-dependent ones, whose energies scale according to r_0^{-2}. In the unitary limit $a \to \infty$, the energy of universal state tends to zero, whereas the energy of potential-dependent state remains finite.

The three-body bound states for two-component fermions in the limit of the infinite two-body scattering length were considered in [44, 45]. The interaction between different particles was taken as the Gaussian potential, whose parameters were adjusted to provide $a \to \infty$.

It was found that in the zero interaction-range limit the $L^P = 1^-$ three-body bound state arises for mass ratio above ≈ 12.314. This value is close to $m/m_1 \approx 12.31310$ determined from the condition $\gamma_1(0) = 1/2$, which means that above this mass ratio the first-channel effective potential in (3) $\left(\gamma_1^2 - 1/4\right)/\rho^2$ becomes attractive at small ρ. A similar problem was considered in [46], in which the potential-dependent three-body bound states were found to exist at least for $m/m_1 > 13$ and two forms of the two-body potential.

IV. CROSSOVER AT THE CRITICAL MASS RATIO

As discussed in Section II, in the zero-range limit the three-body properties are essentially different for mass ratio below and above the critical value μ_c, e. g., a number of bound states abruptly increases from two to infinity [15, 28–30]. Thus, one naturally needs to describe a crossover with increasing m/m_1 beyond the critical value μ_c.

Recently, to study the crossover at μ_c, the dependence of the three-body bound state energies on mass ratio and the additional short-range three-body potential was calculated by using the momentum-space integral equations in [35]. For mass ratio above μ_c the three-body potential is necessary to provide unambiguous description of the wave function near the triple collision point. Explicitly, the momentum cutoff in the integral equation is introduced in paper [35] by using the dimensionless parameter Λ. Analysing the dependence of three-body energies in the plane of two parameters m/m_1 and Λ, it was found that the universal bound states found in [15, 28–30] are located in the region $\mu_1 \le m/m_1 \le \mu_c$ and $\Lambda^{-1} \to 0$, whereas the Efimov states are located in the region $m/m_1 > \mu_c$ and $\Lambda^{-1} \to 0$. These two regions are joined by the crossover area, where the bound-state energies depend on both m/m_1 and Λ and are almost independent of the particular choice of the three-body potential.

V. ZERO-RANGE MODEL IN THE FEW-FERMION PROBLEM

Mathematical aspects of application of the zero-range-interaction model to few two-component fermions were investigated, e. g., in papers [18, 47–49]. In this respect, note the paper [47], in which it was shown that the zero-range model for three and four fermions can be correctly defined for sufficiently small mass ratios. Similar statement was made for an ar-

bitrary number of fermions in [49]. Among discussions of properties of three two-component fermions, one should mention a paper [18], in which the original Efimov's statement was proved, viz., the energy spectrum for the quantum numbers $L^P = 1^-$ becomes unbound from below for $m/m_1 > \mu_c$ (corresponding to $\gamma_1^2(0) < 0$). Recently, it was shown [48] that the three-fermion Hamiltonian is ambiguous in the zero-range limit if mass ratio exceeds 12.31310 (corresponding to $\gamma_1(0) \le 1/2$). It is important to note simple considerations [50], which indicates an existence of two square-integrable solutions in the vicinity of the triple-collision point $\rho \to 0$ for mass ratio in the interval $8.619 \le m/m_1 \le 13.607$ (corresponding to $1 \ge \gamma_1(0) \ge 0$). Therefore, for mass ratio that belongs to this interval, special care is needed to describe three two-component fermions.

VI. CONCLUDING REMARKS

An appearance of three-atomic molecules containing two heavy and one light particles crucially determines the equilibrium states and dynamics of both fermionic and fermionic-bosonic mixtures. One of interesting examples is the mixture of strontium and lithium isotopes. As for the ^{7}Li – ^{87}Sr mixture mass ratio ($m/m_1 \approx 12.4$) gets between μ_1 and μ_2 (at which the first and second bound states emerge), one expects that there is exactly one P-wave bound state of ^{7}Li$\,^{87}$Sr$_2$ molecule. Calculations [15] predict energy of this molecule about -1.793 (recall that the binding energy of the diatomic ^{7}Li$\,^{87}$Sr molecule is taken as energy unit). For the ^{6}Li–^{87}Sr mixture mass ratio ($m/m_1 \approx 14.5$) slightly exceeds μ_c, which means that there are at least two P-wave bound states of the molecule ^{7}Li$\,^{87}$Sr$_2$, whose energies should be slightly below -5.895 and -1.138 (in units of ^{6}Li$\,^{87}$Sr binding energy). Furthermore, as $m/m_1 > \mu_c$, one expects that a number of bound states and their energies depend on the details of the interactions.

Besides influence of the three-body bound states on the properties of the two-component ultra-cold gases, significance of the three-body resonances was emphasised in [31]. Furthermore, for an experimentally interesting case of ^{6}Li–^{40}K fermionic mixture mass ratio $m/m_1 \approx 6.64$ is close to the value $\mu_1 \approx 8.17$ (at which the bound state arises), a role of the P-wave three-body resonance in scattering of ^{6}Li off the ^{6}Li$\,^{40}$K molecule was elucidated in [31–34]. It is worthwhile to mention also ^{173}Yb and ^{23}Na fermion-boson mixture, for which the P-wave resonance should be taken into account as $m/m_1 \approx 7.52$ is even closer to

μ_1. Possible influence of the D-wave resonance can be considered for ^{133}Cs and ^{6}Li mixture, for which $m/m_1 \approx 22.17$ is just below $\mu_1(2) \approx 22.34$ (corresponding to appearance of the D-wave three-body bound state).

The effect of the P-wave three-body bound state in the problem of a light impurity atom immersed in Fermi gas was considered in [9]. By using the variational method, the ground-state properties were determined as a function of mass ratio and dimensionless scattering length and corresponding phase diagram was constructed. It was shown that for a sufficiently large mass ratio ($m/m_1 > 7$) the formation of a three-body molecule is energetically more preferable than a two-body molecule or polaron. As expected, on the phase diagram the boundary between regions corresponding to two- and three-body molecules in the low-density limit tends to the pure three-body result μ_1. Furthermore, the calculated phase diagram shows that with increasing Fermi-gas density the formation of three-atomic molecules becomes more favourable than two-atomic ones.

There is only scarce information about the properties of four two-component fermions. One of the principle results on three identical fermions and distinct particle was obtained in [51], where it was shown that the four-body $L^P = 1^+$ spectrum is not bounded from below for $m/m_1 \ge 13.384$. This means that the four-body Efimov effect takes place in the interval $13.384 \le m/m_1 \le 13.607$. Note that this peculiarity is not present for other values of total angular momentum and parity ($L^P = 0^+$, 1^-). Similar to the three-body problem [15, 28–30], one supposes that the universal four-body bound states could exist below the four-body critical mass ratio $m/m_1 \approx 13.384$. Till now, there is only the calculation [52], which indicates existence of $L^P = 1^+$ bound state of three identical fermions and a distinct particle for $m/m_1 > 9.5$. Besides, one should note the interesting result on the four-body scattering problem [32], where the scattering length for two colliding ^{6}Li ^{40}K molecules was calculated.

Despite a marked success in describing a few two-component fermions, there are still many problems to be solved. In particular, some questions naturally arise in consideration of the potential-dependent states, at least in the limit $a \to \infty$. It is of interest to determine the smallest mass ratio, below which three two-component fermions are not bound by any two-body potential, and to find explicitly the corresponding potential. Likewise, it is desirable to find a maximum number of bound states, which could arise for mass ratio increasing up to μ_c, and to determine the corresponding potential.

In order to elucidate the crossover from finite to infinite number of bound states for

m/m_1 around μ_c, it seems to be important to take into account the energy dependence on the potential range r_0. In a similar way, it is of interest to trace a fate of the potential-dependent states (e.g., in the unitary limit $a \to \infty$) for m/m_1 increasing across the critical value μ_c. In view of the above discussion in Section V on application of the zero-range model in a few-fermion problem, a special care on the asymptotic dependence of the solution near the triple collision point is needed to provide a complete description in the interval $8.62 \le m/m_1 \le 13.607$.

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